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Magnetohydrodynamics Transient Free Convection in Open-Ended Vertical Annuli

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Using Green's function method, analytical solutions for transient magnetohydrodynamics (MHD) fully developed natural convection in open-ended vertical concentric annuli are presented for four fundamental cases of boundary conditions. Expressions for transient fully developed volumetric flow rate, mixing cup temperature, and local Nusselt number are given for each fundamental case. These fundamental solutions may be used to obtain solutions satisfying more general thermal boundary conditions. Also, in a manner similar to the annular geometry, analytical solution for transient MHD fully developed natural convection in an open-ended vertical tube are presented for two fundamental cases of boundary conditions.

Nomenclature

a = local heat transfer coefficient based on the area of heat transfer surface $q/(T_w - T_0) = \pm k(\partial T/\partial r)_w/(T_w - T_0)$, minus and plus signs apply, respectively, for heating and cooling at the inner boundary and vice versa at the outer boundary
B, B_0 = magnetic flux density
b = annular gap width, $r_2 - r_1$
c = specific heat of fluid at constant pressure
D = equivalent (hydraulic) diameter of annulus, $2b$
D_w = diameter of heat transfer boundary
E = induced electric field
F = dimensionless volumetric flow rate, $f/(\pi l \gamma Gr^*)$
f = volumetric flow rate, $\int_{r_1}^{r_2} 2\pi r u \, \mathrm{d}r$
Gr = Grashof number, $\mp g\beta(T_w - T_0)D^3/\gamma^2$ in the case of an isothermal boundary or $\mp g\beta q D^4/2k\gamma^2$ in the case of uniform heat flux heat transfer boundary; the plus and minus signs apply to upward (heating) and downward (cooling) flows, respectively, thus, Gr is a positive number in both cases
Gr^* = modified Grashof number, DGr/l
g = gravitational body force per unit mass
Ha = Hartmann number, $\sqrt{\sigma r_2^2 B_0^2/\mu}$
J_0, J_1, Y_0, Y_1 = Bessel functions
k = thermal conductivity
L = dimensionless height of annulus, $1/Gr^*$
l = height of annulus
N = annulus radius ratio, r_1/r_2
Nu = local Nusselt number, $|a|D/k$
$\overline{Nu}$ = average Nusselt number, $\int_0^l Nu \, \mathrm{d}z/l$
P = dimensionless pressure defect at any point, $p' r_2^4/\rho_0 l^2 \gamma^2 Gr^{*2}$
Pr = Prandtl number, γ/α
p = pressure of fluid inside the channel at any cross section
p' = pressure defect at any point, $p - p_s$
p_s = hydrostatic pressure, $\rho_0 g z$
q = heat flux at the heat transfer surface, $\mp k(\partial T/\partial r)_w$, where the minus and plus signs are, respectively, for heating and cooling in case I; these signs should be reversed in case O
R = dimensionless radial coordinate, r/r_2
r = radial coordinate
r_1, r_2 = inner and outer radii of annulus
T = temperature at any point
T_m = mixing cup temperature over any cross section, $\int_{r_1}^{r_2} ruT \, \mathrm{d}r/\int_{r_1}^{r_2} ru \, \mathrm{d}r$
T_w = temperature of heat transfer boundary
T_0 = temperature at the annulus entrance
t = time
U = dimensionless volume averaged axial velocity, $ur_2^2/(l\gamma Gr^*)$
u = axial velocity
Z = dimensionless axial coordinate, $z/(lGr^*)$
z = axial coordinate
α = thermal diffusivity, $k/\rho c$
β = volumetric coefficient of thermal expansion
γ = kinematic viscosity of fluid, μ/ρ_0
θ = dimensionless temperature, $(T - T_0)/(T_w - T_0)$ in the case of an isothermal heat transfer boundary and $(T - T_0)/(qD/2k)$ for uniform heat flux boundary and, thus, it is positive for both heating (upward) and cooling (downward) flows
θ_m = dimensionless mixing cup temperature, $(T_m - T_0)/(T_w - T_0)$ in the case of an isothermal heat transfer boundary and $(T_m - T_0)/qD/2k)$ for uniform heat flux boundary
θ_w = dimensionless temperature of heat transfer boundary
μ = dynamic viscosity of fluid
ρ = fluid density at T, $\rho_0[1 - \beta(T - T_0)]$
ρ_0 = fluid density at T_0
σ = electrical conductivity of the fluid
τ = dimensionless time, $tk/\rho c r_2^2$

Subscripts

r = radial coordinate
z = axial coordinate
θ = θ coordinate

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Introduction

MAGNETOHYDRODYNAMICS (MHD) free-convection flow of an electrically conducting fluid in different geometries is of considerable interest because of its frequent occurrence in industrial and technological applications. Examples of these applications is in power generators, nuclear reactors, and in MHD accelerators.

Many papers concerned with the problem of MHD free-convection flow have been published in the literature. As an example, free convection flow of an electrically conducting fluid past over or into different geometries is investigated by different researchers.[1–5]

To the authors' knowledge, no previous study has considered the transient behavior of MHD free-convection flow in open-ended vertical annuli. Our aim is to consider the transient MHD free-convection problems in vertical annuli to improve our physical understanding of their hydrodynamic and thermal behaviors.

Governing Equations and Boundary Conditions

We consider unsteady laminar fully developed free-convection flow of an electrically conducting and incompressible viscous fluid inside an open-ended vertical concentric annulus of a finite length (l), immersed in a stagnant fluid of infinite extent maintained at a constant temperature T_0. The walls of the annulus are assumed to be nonconducting. A uniform magnetic field is assumed to be applied in the radial direction. The impressed electrical field is assumed to be zero and the induced magnetic field of the flow is negligible in comparison with the applied one, which corresponds to a very small magnetic Reynolds number.[6] Figure 1 shows the physical situation in which at least one of the channel walls is heated or cooled either isothermally or at a constant wall heat flux, so that its temperature, i.e., temperature of the inner surface of the outer cylinder or that of the outer surface of the inner cylinder, is different from the ambient temperature T_0. Because of fully developed flow assumptions, the fluid enters the part under consideration of the annular passage with an axial velocity profile that remains invariant in the entire channel, i.e., $\partial u/\partial z = 0$. The fluid is assumed to be Newtonian, enters the channel at the ambient temperature T_0, and obeys the Boussinesq approximation, according to which its density is constant except in the gravitational term of the vertical momentum equation. Axial symmetry is assumed and viscous dissipation and internal heat generation are absent. Also, as a result of the zero impressed electrical field and negligible ohmic heating, electromagnetic work does not ensue.

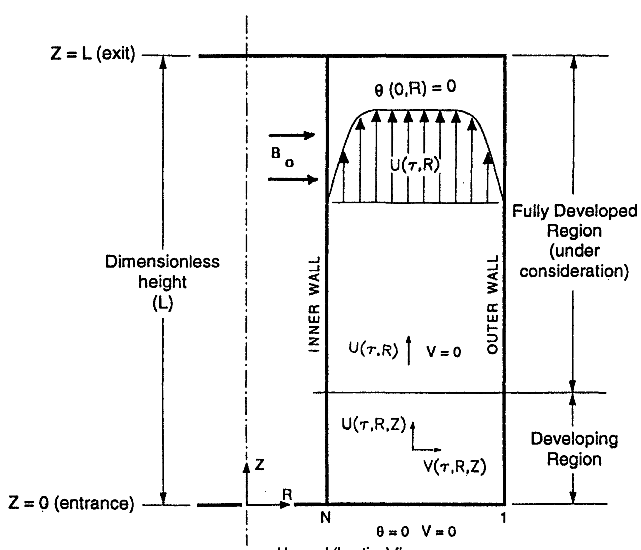

Fig. 1 Schematic diagram.

Under the previously mentioned assumptions and using the dimensionless parameters given in the Nomenclature, the equations of continuity, motion, and energy reduce to the following two simultaneous nondimensional equations:

$$\frac{1}{Pr}\frac{\partial U}{\partial \tau} = -\frac{\partial P}{\partial Z} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right) - Ha^2 U + \frac{\theta}{16(1-N)^4} \quad (1)$$

$$\frac{\partial \theta}{\partial \tau} + PrU\frac{\partial \theta}{\partial Z} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \theta}{\partial R}\right) \quad (2)$$

where Ha is the Hartmann number that measures the strength of the magnetic field.

In Eq. (1), the imposed radial magnetic field is assumed to be steady and uniform. This is a common practice in the literature[2,7] and it is a good approximation when the gap width is small compared with either the inner or outer radius of the annular. In this case, $\partial B/\partial r \approx 0$, which implies that B = const = B_0. With this assumption it is not necessary to simplify other terms in Eqs. (1) and (2) to become as those for parallel plate channels. In addition, both induced electrical and magnetic fields effects on Eq. (1) are neglected. The assumption of zero-induced electric field can be acheived by short-circuiting the walls of the annulus. This assumption may be clarified by refering to Maxwell's equations:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \approx 0 \quad (3)$$

from which

$$(\nabla \times \boldsymbol{E})_z = \frac{\partial E_\theta}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = \frac{\partial E_\theta}{\partial r} \approx 0 \quad (4)$$

Equation (4) implies that $E_\theta \approx f(\theta)$ = const because of symmetry. If one of the walls of the annulus is short-circuited, $E_\theta = 0$ inside this wall. This implies that $E_\theta = 0$ in the fluid because E_θ is continuous at the wall–fluid interface. On the other hand, neglecting the induced magnetic field is a common practice in the literature.[2,7] The last term in Eq. (1) accounts for the natural convection effects and makes the momentum equation coupled with the energy equation.

Two initial conditions and four boundary conditions are therefore needed to obtain a solution for Eqs. (3) and (4). The two initial conditions are

$$U(0, R, Z) = \theta(0, R, Z) = 0 \quad (5)$$

The two boundary conditions related to U are

$$U(\tau, 1, Z) = U(\tau, N, Z) = 0 \quad (6)$$

On the other hand, there are many possible thermal boundary conditions applicable to the annular configuration. Reynolds et al.[8] defined four fundamental boundary conditions for the annular geometry that produces four fundamental solutions to the energy equation when it becomes linear. For the sake of completeness, these fundamental solutions are stated in the following text.

1) Fundamental solutions of the first kind, which satisfy the boundary conditions of a temperature step change at one wall, the opposite wall being kept isothermal at the inlet fluid temperature. Using the present notation, this corresponds to $\theta = 1$ at one wall and $\theta = 0$ at the opposite wall for $\tau > 0$, where the boundaries are kept at the inlet fluid temperature; $\theta = 0$ for $\tau \leq 0$ for all cases.

2) Fundamental solutions of the second kind, which satisfy the boundary conditions of a step change in heat flux at one wall, the opposite wall being adiabatic. Using the present notation, this corresponds to $\partial\theta/\partial R = -1/(1 - N)$ at the inner

wall and $\partial\theta/\partial R = 0$ at the outer wall, or $\partial\theta/\partial R = 0$ at the inner wall and $\partial\theta/\partial R = 1/(1 - N)$ at the outer wall for $\tau > 0$.

3) Fundamental solutions of the third kind, which satisfy the boundary conditions of a temperature step change at one wall, the opposite wall being adiabatic. This corresponds to $\theta = 1$ at one wall and $\partial\theta/\partial R = 0$ at the opposite wall for $\tau > 0$.

4) Fundamental solutions of the fourth kind, where a step change in heat flux at one wall is applied while the opposite wall is kept isothermal at the inlet fluid temperature. This corresponds to $\partial\theta/\partial R = -1/(1 - N)$ at the inner wall, whereas $\theta = 0$ at the outer wall or $\theta = 0$ at the inner wall, and $\partial\theta/\partial R = 1/(1 - N)$ at the outer wall for $\tau > 0$.

With any of the previously mentioned boundary conditions, the boundary opposite to that maintained adiabatic, i.e., $\partial\theta/\partial R = 0$, or isothermal, i.e., $\theta = 0$, is termed the heat transfer boundary (even though there is transfer of heat through a boundary maintained at $\theta = 0$). For each of the previous fundamental solutions, two cases are considered, namely, case (I), in which the heat transfer boundary is at the inner wall and case (O) in which the heat transfer boundary is at the outer wall. The aim of the present paper is to obtain the previously mentioned four fundamental solutions.

General Analysis

Substituting θ from Eq. (1) into Eq. (2), we obtain

$$\left[\frac{\partial^2 U}{\partial \tau^2} + Pr\frac{\partial^2 P}{\partial \tau \partial Z} - \frac{Pr}{R}\frac{\partial}{\partial R}\left(R\frac{\partial^2 U}{\partial \tau \partial R}\right) + PrHa^2\frac{\partial U}{\partial \tau}\right] + Pr^2 U\frac{\partial^2 P}{\partial Z^2} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\left\{\frac{\partial^2 U}{\partial \tau \partial R} - Pr\frac{\partial}{\partial R} \times \left[R\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right)\right] + PrHa^2\frac{\partial U}{\partial R}\right\}\right) \tag{7}$$

A solution of Eq. (7) in the form $U = U(\tau, R)$ is only possible if

$$\frac{\partial^2 P}{\partial \tau \partial Z} = \bar{\gamma}(\tau) \tag{8}$$

$$\frac{\partial^2 P}{\partial Z^2} = \bar{\alpha}(\tau) \tag{9}$$

Integrating Eq. (8) with respect to time yields

$$\frac{\partial P}{\partial Z} = \bar{\beta}(\tau) + E(Z) \tag{10}$$

from which

$$\frac{\partial^2 P}{\partial Z^2} = E'(Z) = \bar{\alpha}(\tau) \tag{11}$$

But $E'(Z)$ is independent of time, and as a result

$$\frac{\partial^2 P}{\partial Z^2} = \bar{\alpha} \tag{12}$$

where $\bar{\alpha}$ is constant. Equation (12) gives the solution for P as

$$P = 0.5\bar{\alpha}Z^2 + \bar{\beta}Z + \bar{\beta}Z + \bar{\delta}(\tau) \tag{13}$$

Applying the conditions for an open-ended channel, that $P = 0$ at both inlet and exit, i.e., at $Z = 0$ and L, gives

$$P = 0.5\bar{\alpha}Z(Z - L) \tag{14}$$

From Eq. (1) we have

$$\frac{\partial \theta}{\partial Z} = 16\bar{\alpha}(1 - N)^4 \tag{15}$$

which means that, for a given R in a given annulus, the dimensionless temperature θ varies linearly with the axial distance Z. This implies that the assumption of a hydrodynamically fully developed free convection flow should necessarily mean that the flow is also thermally fully developed, regardless of the value of the Prandtl number (Pr). As a result of the conclusion that α is constant, Eq. (7) is reduced to

$$\left[\frac{\partial^2 U}{\partial \tau^2} - \frac{Pr}{R}\frac{\partial}{\partial R}\left(R\frac{\partial^2 U}{\partial \tau \partial R}\right) + PrHa^2\frac{\partial U}{\partial \tau}\right] + \bar{\alpha}Pr^2 U = \frac{1}{R}\frac{\partial}{\partial R}\left(R\left\{\frac{\partial^2 U}{\partial \tau \partial R} - Pr\frac{\partial}{\partial R}\left[R\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right)\right] + PrHa^2\frac{\partial U}{\partial R}\right\}\right) \tag{16}$$

The governing Eqs. (1) and (2) can be simplified if one of the two annulus boundaries is kept isothermal. To satisfy this boundary condition, θ must, in this particular case, be independent of Z. Thus, it is concluded that α must, in such a case, equal zero. Therefore, Eqs. (14) and (15) reduce, in this case, to the following equations, respectively,

$$P = 0 \tag{17}$$

$$\frac{\partial \theta}{\partial Z} = 0 \tag{18}$$

Equation (18) states that, in a case with an isothermal boundary, the fully developed temperature profile is a function of R and τ only. On the other hand, Eq. (17) states that the fully developed pressure inside an open-ended annulus of an isothermal boundary is equal to the hydrostatic pressure, at the same elevation, outside the annulus. This implies that, in such a fully developed case with an isothermal boundary, there would be no pressure drop caused by fluid viscous or magnetic drag because these are just offset by the buoyancy driving force.

If the two governing Eqs. (1) and (2) are solved for the velocity and temperature profiles (U and θ), then the following useful parameters can be evaluated.

The dimensionless volumetric flow rate (F) can be evaluated from the following equation:

$$F = 2\int_N^1 RU\,\mathrm{d}R \tag{19}$$

Because, for a fully developed flow, U is a function of R and τ only, it follows that the definite integral on the right-hand side of Eq. (19) and, hence, F are functions of τ regardless of the value of the axial coordinate Z; i.e., they are not related to the value of the annulus height.

The dimensionless mixing cup temperature is given by

$$\theta_m = \frac{\int_N^1 RU\theta\,\mathrm{d}R}{\int_N^1 RU\,\mathrm{d}R} \tag{20}$$

Using the dimensionless parameters given in the Nomenclature, the following expressions for the local Nusselt number can easily be obtained:

For a uniform wall temperature (UWT) boundary conditions

$$Nu = \pm 2(1 - N)\left(\frac{\partial \theta}{\partial R}\right)_w \tag{21}$$

and for a uniform heat flux (UHF) boundaries

$$Nu = \pm \frac{2(1 - N)}{\theta_w}\left(\frac{\partial \theta}{\partial R}\right)_w = \frac{2}{\theta_w} \tag{22}$$

where the minus and plus signs apply, respectively, for cases (I) and (O) when there is heating and vice versa when there is cooling.

From Eq. (1) it can be seen that $(\partial\theta/\partial R)$ is a function of R and τ only, which is dependent on the fully developed axial velocity profile (U), i.e., it is independent of Z. Hence, for a case with a UWT boundary, Eq. (21) shows that the fully developed local Nusselt number is a function of time only. Consequently, the fully developed average Nusselt number is, in this case UWT, independent of annulus height L. On the other hand, for boundary conditions other than UWT, provided that the flow is hydrodynamically fully developed, Eq. (15) shows that the temperature varies linearly with Z. Hence, Eq. (22) shows that the fully developed local Nusselt number, for this case, varies hyperbolically with Z.

It is important to mention here that, to maintain the validity of the hydrodynamics fully developed flow assumption, the only thermal boundary conditions accepted, other than the UWT boundary conditions, are those that vary linearly with Z. As a result, all of the problems that include boundary conditions other than the UWT can be considered as fundamental problems of the second kind.

Fundamental Solutions

If at least one of the two annulus boundaries is kept isothermal, Eqs. (1) and (2) are reduced to

$$\frac{1}{Pr}\frac{\partial U}{\partial \tau} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right) - Ha^2 U + \frac{\theta}{16(1 - N)^4} \tag{23}$$

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \theta}{\partial R}\right) \tag{24}$$

Equation (24) assumes a solution in the form

$$\theta(\tau, R) = \theta_1(\tau, R) + \theta_2(R) \tag{25}$$

where $\theta_2(R)$ accounts for the nonhomogeneousity in the boundary conditions. The solution of the homogeneous part obtained by the separation of variables as

$$\theta_1(\tau, R) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 \tau}[Y_0(\lambda_n R) - C_n J_0(\lambda_n R)] \tag{26}$$

$$A_n = -\frac{\int_N^1 \theta_2(R) R [Y_0(\lambda_n R) - C_n J_0(\lambda_n R)]\, \mathrm{d}R}{\int_N^1 R[Y_0(\lambda_n R) - C_n J_0(\lambda_n R)]^2\, \mathrm{d}R} \tag{27}$$

where $\theta_2(R)$, C_n, and λ_n depend on the kind of fundamental case we have.

The solution of Eq. (23) is obtained by the application of Green's function approach. Let $U(\tau, R)$ be related to $U'(\tau, R)$ by

$$U(\tau, R) = U'(\tau, R)\exp(-PrHa^2\tau) \tag{28}$$

Substitute Eq. (28) into Eq. (23) to obtain

$$\frac{1}{Pr}\frac{\partial U'}{\partial \tau} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U'}{\partial R}\right) + \frac{\theta}{16(1 - N)^4}\exp(PrHa^2\tau) \tag{29}$$

For generality, assume boundary conditions in the forms

$$U'(\tau, 1) = f_1(\tau, R), \qquad U'(\tau, N) = f_2(\tau, R) \tag{30}$$

and initial condition

$$U'(0, R) = \bar{F}(R) \tag{31}$$

where $f_i(\tau, R)(i = 1, 2)$ and $\bar{F}(R)$ have zero values for our particular case.

The desired Green's function is obtained from the solution of the homogenous version of the problem defined in Eqs. (29–31) as

$$G(R, \tau|R', \tau') = \sum_{m=1}^{\infty} \frac{e^{-\beta_m^2 Pr(\tau-\tau')}}{S}\phi_m(R)\phi_m(R') \tag{32}$$

where

$$\phi_m(R) = Y_0(\beta_m R) - b_m J_0(\beta_m R) \tag{33}$$

$$S = \int_N^1 R[Y_0(\beta_m R) - b_m J_0(\beta_m R)]^2\, \mathrm{d}R, \qquad b_m = \frac{Y_0(\beta_m)}{J_0(\beta_m)} \tag{34}$$

and the eigenvalues β_m are positive roots of

$$Y_0(\beta_m N) - b_m J_0(\beta_m N) = 0 \tag{35}$$

Then the solution of the nonhomogeneous problem [Eqs. (29–31)], in terms of the previous Green's function is given as[9]

$$\begin{aligned} U'(\tau, R) = {} & \int_N^1 R' G(R, \tau|R', \tau')|_{\tau'=0}\, \bar{F}(R')\, \mathrm{d}R' \\ & + \frac{Pr}{16(1 - N)^4}\int_{\tau'=0}^{\tau} \mathrm{d}\tau' \int_N^1 R' G(R, \tau|R', \tau') \\ & \times e^{(PrHa^2\tau')}\theta(R', \tau')\, \mathrm{d}R' \\ & - Pr\int_{\tau'=0}^{\tau} \mathrm{d}\tau' \sum_{i=1}^{2}\left[R'\frac{\partial G}{\partial R'}\right]_{R'=R_i} f_i(R', \tau') \end{aligned} \tag{36}$$

Fundamental Solutions of the First Kind

In this case, the two boundaries of the annulus are kept isothermal, one of which is at the inlet ambient fluid temperature T_0 ($\theta = 0$), whereas the opposite boundary is at a higher or a lower temperature. The following thermal boundary conditions can be applied:

Case (I): temperature step at the inner wall, whereas the outer wall is kept at the ambient temperature, i.e.,

$$\theta(\tau, 1) = 0, \qquad \theta(\tau, N) = 1, \qquad \tau > 0 \tag{37}$$

Case (O): temperature step at the outer wall, whereas the inner wall is kept at the ambient temperature, i.e.,

$$\theta(\tau, N) = 0, \qquad \theta(\tau, 1) = 1, \qquad \tau > 0 \tag{38}$$

The evaluation of the required parameters is as follows.
Case (I): the eigenvalues λ_n are the roots of

$$Y_0(\lambda_n N) - C_n J_0(\lambda_n N) = 0 \quad (39)$$

where $C_n = Y_0(\lambda_n)/J_0(\lambda_n)$, and

$$\theta_2(R) = \ell n\, R/\ell n\, N \quad (40)$$

A_n can be evaluated from Eq. (27) as

$$A_n = (N/\lambda_n M)[Y_1(\lambda_n N) - C_n J_1(\lambda_n N)] \quad (41)$$

where

$$M = \tfrac{1}{2}[Y_1(\lambda_n) - C_n J_1(\lambda_n)]^2 - (N^2/2)[Y_1(\lambda_n N) - C_n J_1(\lambda_n N)]^2 \quad (42)$$

Substituting $\theta(\tau, R)$ into Eq. (36) with $f_1(\tau, R)$, $f_2(\tau, R)$, and $\bar{F}(R)$ equal to zero, and using Eq. (28) we get

$$U(\tau, R) = \frac{1}{16(1-N)^4} \times \left(\sum_{m=1}^{\infty} B_m\{1 - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\}\phi_m(R) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\{e^{-\lambda_n^2\tau} - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\}\phi_m(R)\right) \quad (43)$$

where

$$B_m = -\frac{N[Y_1(\beta_m N) - b_m J_1(\beta_m N)]}{S(\beta_m^3 + \beta_m Ha^2)} \quad (44)$$

$$B_{mn} = \begin{cases} 0 & \text{if} \quad \beta_m \neq \lambda_n (m \neq n) \\ \dfrac{A_n}{\beta_m^2 + Ha^2 - (\lambda_n^2/Pr)} & \text{if} \quad \beta_m = \lambda_n (m = n) \end{cases} \quad (45)$$

and from Eq. (34)

$$S = \tfrac{1}{2}[Y_1(\beta_m) - b_m J_1(\beta_m)]^2 - (N^2/2)[Y_1(\beta_m N) - b_m J_1(\beta_m N)]^2 \quad (46)$$

The volume flow rate F and the dimensionless mixing cup temperature θ_m, defined by Eqs. (19) and (20) are readily calculated using solutions for U and θ. The results are as follows:

$$F(\tau) = \frac{1}{8(1-N)^4}\left(\sum_{m=1}^{\infty} C_m\{1 - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\{e^{-\lambda_n^2\tau} - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\}\right) \quad (47)$$

where

$$C_m = \frac{B_m}{\beta_m}\{[Y_1(\beta_m) - b_m J_1(\beta_m)] - N[Y_1(\beta_m N) - b_m J_1(\beta_m N)]\} \quad (48)$$

$$C_{mn} = \frac{B_{mn}}{\beta_m}\{[Y_1(\beta_m) - b_m J_1(\beta_m)] - N[Y_1(\beta_m N) - b_m J_1(\beta_m N)]\} \quad (49)$$

$$\theta_m(\tau) = \frac{1}{8F(1-N)^4} \times \left(\sum_{m=1}^{\infty} E_m\{1 - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\}(1 - e^{-\beta_m^2\tau}) + \sum_{m=1}^{\infty} E'_m\{e^{-\beta_m^2\tau} - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\}(1 - e^{-\beta_m^2\tau}\tau)\right) \quad (50)$$

where

$$E_m = \frac{N^2[Y_1(\beta_m N) - b_m J_1(\beta_m N)]^2}{S[\beta_m^4 + \beta_m^2 Ha^2]} \quad (51)$$

$$E'_m = -\frac{N^2[Y_1(\beta_m N) - b_m J_1(\beta_m N)]^2}{S[\beta_m^4 + \beta_m^2 Ha^2 - (\beta_m^4/Pr)} \quad (52)$$

It may be worth mentioning that, in the present case of isothermal boundaries, the temperature $\underline{\theta}$ (and, hence, θ_m), does not vary with axial distance Z. Thus, $\overline{\theta_m} = \theta m$. This means that the heat transferred to/from the fluid through the two boundaries of the annulus, under the fully developed flow conditions, does not affect the fluid bulk temperature because they are equal and opposite (so that the fully developed conditions can be achieved in such a case). Expressions for the fully developed Nusselt number (local and also average) are obtained after getting the temperature gradient at the walls from Eq. (25) and then substituting into Eq. (21). The value of Nu on the inner wall is given as

$$Nu_I(\tau) = \pm 2(1 - N)\left(\left\{\sum_{n=1}^{\infty} \lambda_n A_n e^{-\lambda_n^2\tau}[-Y_1(\lambda_n N) + C_n J_1(\lambda_n N)]\right\} + \frac{1}{N\,\ell n\, N}\right) \quad (53)$$

where the minus and plus signs apply, respectively, for heating and cooling.

Case (O): here, different parameters are given as

$$\theta_2(R) = 1 - \frac{\ell n\, R}{\ell n\, N}, \qquad A_n = -\frac{1}{\lambda_n M}[Y_1(\lambda_n) - C_n J_1(\lambda_n)] \quad (54)$$

where the eigenvalues λ_n are still given by Eq. (39), and M is given by Eq. (42).

Also, the velocity profile is given as in Eq. (43) with

$$B_m = \frac{Y_1(\beta_m) - b_m J_1(\beta_m)}{S(\beta_m^3 + \beta_m Ha^2)} \quad (55)$$

where B_{mn} and S are given as in Eqs. (45) and (46), respectively.

The volume flow rate F, C_m, and C_{mn} are given as in Eqs. (47), (48), and (49), respectively.

The dimensionless mixing cup temperature θ_m is given as in Eq. (50), but with

$$E_m = \frac{[Y_1(\beta_m) - b_m J_1(\beta_m)]^2}{S[\beta_m^4 + \beta_m^2 Ha^2]}$$

$$E'_m = -\frac{[Y_1(\beta_m) - b_m J_1(\beta_m)]^2}{S[\beta_m^4 + \beta_m^2 Ha^2 - (\beta_m^4/Pr)]} \quad (56)$$

The expression for Nu on the outer wall is given as

$$Nu_o(\tau) = \pm 2(1 - N) \times \left(\left\{ \sum_{n=1}^{\infty} \lambda_n A_n e^{-\lambda_n^2 \tau}[-Y_1(\lambda_n) + C_n J_1(\lambda_n)] \right\} - \frac{1}{\ell n N} \right) \quad (57)$$

A sample of the results is plotted in Figs. 2 and 3. These figures represent the thermal and the hydrodynamics transient behavior of the fluid for case (I). It is clear from Fig. 2 that the peak of the velocity profile is shifted to the heating wall. It will be shifted to the inner wall in case (I) and to the outer wall in case (O). However, if both walls have the same thermal boundary conditions, then the peak will be shifted to the inner wall.

Fundamental Solutions of the Second Kind

In this case, one of the annulus boundaries is maintained at a constant heat flux (q) and the opposite boundary is perfectly insulated. The governing equations in such a case are Eqs. (1) and (2), where $\partial\theta/\partial Z \neq 0$. We are unable to get a closed-form solution for this case.

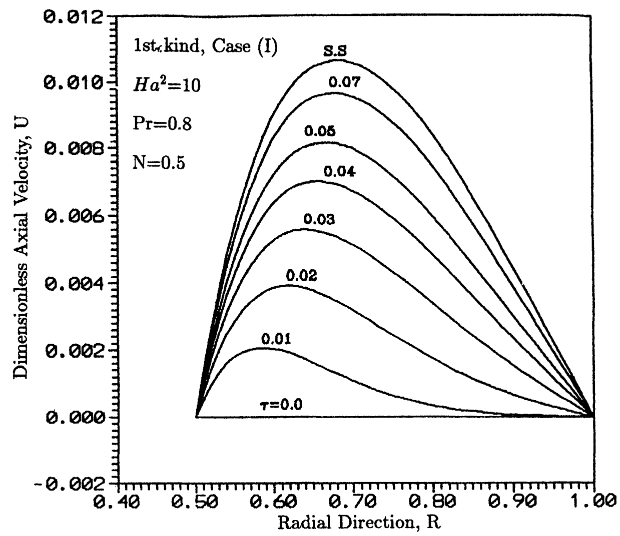

Fig. 2 Transient axial velocity distribution in the radial direction.

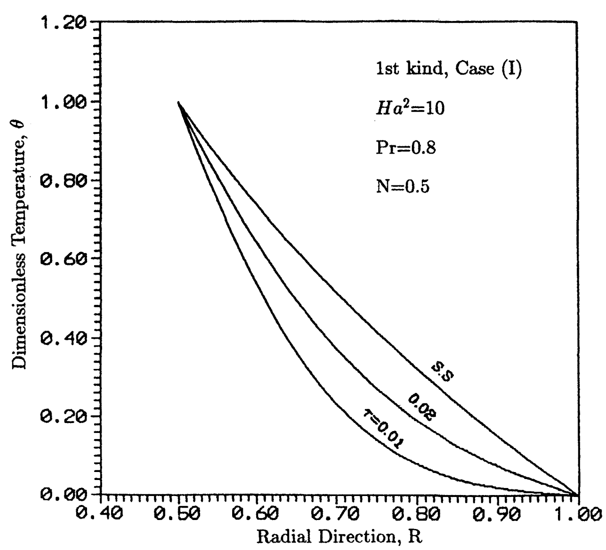

Fig. 3 Transient temperature distribution in the radial direction.

Fundamental Solutions of the Third Kind

In this case, because one of the boundaries is isothermal, Eqs. (23) and (24) are the governing equations subject to the following boundary conditions:

Case (I): temperature step at the inner wall, whereas the outer wall is kept insulated, i.e.,

$$\frac{\partial \theta}{\partial R}(\tau, 1) = 0, \qquad \theta(\tau, N) = 1, \qquad \tau > 0 \quad (58)$$

Case (O): temperature step at the outer wall, whereas the inner wall is kept insulated, i.e.,

$$\frac{\partial \theta}{\partial R}(\tau, N) = 0, \qquad \theta(\tau, 1) = 1, \qquad \tau > 0 \quad (59)$$

The solutions obtained are as follows.

Case (I): the eigenvalues λ_n are the roots of

$$Y_1(\lambda_n) - C_n J_1(\lambda_n) = 0 \quad (60)$$

where $C_n = Y_0(\lambda_n N)/J_0(\lambda_n N)$ and

$$\theta_2(R) = 1, \qquad A_n = (N/\lambda_n M)[Y_1(\lambda_n N) - C_n J_1(\lambda_n N)] \quad (61)$$

where

$$M = \tfrac{1}{2}[Y_0(\lambda_n) - C_n J_0(\lambda_n)]^2 - (N^2/2)[Y_1(\lambda_n N) - C_n J_1(\lambda_n N)]^2 \quad (62)$$

Also, the velocity profile is given in Eq. (43), but with

$$B_m = \frac{[Y_1(\beta_m) - b_m J_1(\beta_m)] - N[Y_1(\beta_m N) - b_m J_1(\beta_m N)]}{S[\beta_m^3 + \beta_m Ha^2]} \quad (63)$$

$$B_{mn} = \frac{A_n[-\beta_m/(\beta_m^2 - \lambda_n^2)][Y_0(\lambda_n) - C_n J_0(\lambda_n)][Y_1(\beta_m) - b_m J_1(\beta_m)]}{S[\beta_m^2 + Ha^2 - (\lambda_n^2/Pr)]} \quad (64)$$

where S is given as in Eq. (46). The volume flow rate F, C_m, and C_{mn} are given as in Eqs. (47), (48), and (49), respectively. The dimensionless mixing cup temperature θ_m is given as

$$\theta_m(\tau) = \frac{1}{8F(1 - N)^4} \left[\sum_{m=1}^{\infty} E_m\{1 - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E'_m\{e^{-\lambda_n^2\tau} - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mn}(e^{-\lambda_n^2\tau} - \exp\{-Pr[\beta_m^2 + Ha^2 + (\lambda_n^2/Pr)]\tau\}) + \sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mni}(e^{-(\lambda_n^2+\lambda_i^2)\tau} - \exp\{-Pr[\beta_m^2 + Ha^2 + (\lambda_n^2/Pr)]\tau\}) \right] \quad (65)$$

where $E_m = C_m$, $E'_m = C'_m$, and

$$E_{mn} = B_m A_n\{-[\beta_m/(\beta_m^2 - \lambda_n^2)]\}[Y_0(\lambda_n) - C_n J_0(\lambda_n)][Y_1(\beta_m) - b_m J_1(\beta_m)] \quad (66)$$

$$E_{mni} = B_{mn}A_i\{-[\beta_m/(\beta_m^2 - \lambda_i^2)]\}[Y_0(\lambda_i) - C_iJ_0(\lambda_i)][Y_1(\beta_m) - b_mJ_1(\beta_m)] \tag{67}$$

The local Nusselt number on the inner wall is given as

$$Nu_i(\tau) = \pm 2(1 - N)\left\{\sum_{n=1}^{\infty} \lambda_n A_n e^{-\lambda_n^2\tau}[-Y_1(\lambda_n N) + C_nJ_1(\lambda_n N)]\right\} \tag{68}$$

Case (O): the eigenvalues λ_n are the roots of

$$Y_1(\lambda_n N) - C_nJ_1(\lambda_n N) = 0 \tag{69}$$

where $C_n = Y_0(\lambda_n)/J_0(\lambda_n)$ and

$$\theta_2(R) = 1, \qquad A_n = (1/\lambda_n M)[C_nJ_1(\lambda_n) - Y_1(\lambda_n)] \tag{70}$$

$$M = \tfrac{1}{2}[Y_1(\lambda_n) - C_nJ_1(\lambda_n)]^2 - (N^2/2)[Y_0(\lambda_n N) - C_nJ_0(\lambda_n N)]^2 \tag{71}$$

$U(\tau, R)$, $F(\tau)$, and $\theta_m(\tau)$ are the same as case (I), but with

$$B_{mn} = \frac{A_n\{[\beta_m N/(\beta_m^2 - \lambda_n^2)]\}[Y_0(\lambda_n N) - C_nJ_0(\lambda_n N)][Y_1(\beta_m N) - b_mJ_1(\beta_m N)]}{S[\beta_m^2 + Ha^2 - (\lambda_n^2/Pr)]} \tag{72}$$

$$E_{mn} = B_mA_n[\beta_m N/(\beta_m^2 - \lambda_n^2)][Y_0(\lambda_n N) - C_nJ_0(\lambda_n N)] \times [Y_1(\beta_m N) - b_mJ_1(\beta_m N)] \tag{73}$$

$$E_{mni} = B_{mn}A_i[\beta_m N/(\beta_m^2 - \lambda_i^2)][Y_0(\lambda_i N) - C_iJ_0(\lambda_i N)] \times [Y_1(\beta_m N) - b_mJ_1(\beta_m N)] \tag{74}$$

The local Nusselt number on the outer wall is given as

$$Nu_o(\tau) = \pm 2(1 - N)\left\{\sum_{n=1}^{\infty} \lambda_n A_n e^{-\lambda_n^2\tau}[-Y_1(\lambda_n) + C_nJ_1(\lambda_n)]\right\} \tag{75}$$

Note that in cases (I) and (O) the value of the Nu on the insulated wall is zero.

Fundamental Solutions of the Fourth Kind

In this case, because one of the boundaries is isothermal, Eqs. (23) and (24) are the governing equations subject to the following boundary conditions:

Case (I): step change in heat flux at the inner wall, whereas the outer wall is isothermal at the inlet fluid temperature, i.e.,

$$\frac{\partial\theta}{\partial R}(\tau, N) = -\frac{1}{1 - N}, \qquad \theta(\tau, 1) = 0, \qquad \tau > 0 \tag{76}$$

Case (O): step change in heat flux at the outer wall, whereas the inner wall is isothermal at the inlet fluid temperature, i.e.,

$$\frac{\partial\theta}{\partial R}(\tau, 1) = \frac{1}{1 - N}, \qquad \theta(\tau, N) = 0, \qquad \tau > 0 \tag{77}$$

The solutions for both cases are given as

Case (I): the eigenvalues λ_n are the roots of

$$Y_1(\lambda_n N) - C_nJ_1(\lambda_n N) = 0 \tag{78}$$

where $C_n = Y_0(\lambda_n)/J_0(\lambda_n)$ and

$$\theta_2(R) = -\frac{N\,\ell n\,R}{1 - N}$$
$$A_n = \frac{N}{\lambda_n^2 M(N - 1)}[Y_0(\lambda_n N) - C_nJ_0(\lambda_n N)] \tag{79}$$

where M is given as in Eq. (71). Also, the velocity profile is given as in Eq. (43), but with

$$B_m = \frac{N^2\,\ell n\,N[Y_1(\beta_m N) - b_mJ_1(\beta_m N)]}{(1 - N)S[\beta_m^3 + \beta_m Ha^2]} \tag{80}$$

where S and B_{mn} are still given as in Eqs. (46) and (72), respectively.

The volume flow rate F, C_m, and C_{mn} are given by Eqs. (47), (48), and (49), respectively. The dimensionless mixing cup temperature θ_m is given as in Eq. (65), but with

$$E_m = \left(\frac{N^2\,\ell n\,N}{1 - N}\right)^2 \frac{[Y_1(\beta_m N) - b_mJ_1(\beta_m N)]^2}{S(\beta_m^4 + \beta_m Ha^2)} \tag{81}$$

$$E'_m = \frac{B_{mn}N^2\,\ell n\,N}{(1 - N)\beta_m}[Y_1(\beta_m N) - b_mJ_1(\beta_m N)] \tag{82}$$

and E_{mn} and E_{mni} are still given as in Eqs. (73) and (74), respectively.

The value of Nu on the inner wall is given as

$$Nu_i(\tau) = 2/\theta(\tau, N) \tag{83}$$

Case (O): the eigenvalues λ_n are the roots of

$$Y_1(\lambda_n) - C_nJ_1(\lambda_n) = 0 \tag{84}$$

where $C_n = Y_0(\lambda_n N)/J_0(\lambda_n N)$ and

$$\theta_2(R) = \frac{\ell n(R/N)}{1 - N}, \qquad A_n = \frac{Y_0(\lambda_n) - C_nJ_0(\lambda_n)}{\lambda_n^2(1 - N)M} \tag{85}$$

where M is given as in Eq. (62). Also, the velocity profile is given as in Eq. (43), but with

$$B_m = -\frac{\ell n\,N[Y_1(\beta_m) - b_mJ_1(\beta_m)]}{(1 - N)S(\beta_m^3 + \beta_m H\alpha^2)} \tag{86}$$

where S and B_{mn} are given as in Eqs. (46) and (64), respectively.

The volume flow rate F, C_m, and C_{mn} are given as in Eqs. (47), (48), and (49), respectively.

The dimensionless mixing cup temperature θ_{mn} is given as in Eq. (65), but with

$$E_m = \left(\frac{\ell n\,N}{1 - N}\right)^2 \frac{[Y_1(\beta_m) - b_mJ_1(\beta_m)]^2}{S(\beta_m^4 + \beta_m^2 Ha^2)} \tag{87}$$

$$E'_m = \frac{B_{mn}}{\beta_m}\left(\frac{-\ell n\,N}{1 - N}\right)[Y_1(\beta_m) - b_mJ_1(\beta_m)] \tag{88}$$

where E_{mn} and E_{mni} are still given as in Eq. (66) and (67), respectively.

The value of Nu on the outer wall Nu_o is given as

$$Nu_o(\tau) = 2/\theta(\tau, 1) \tag{89}$$

The effect of Ha on the radial distribution of the axial velocity is shown in Figs. 4–6 for case (I) of the three fundamental kinds. It is clear from these figures that increasing Ha increases the magnetic retardation force, which leads to a significant reduction in the axial velocity. The effect of small Ha is more significant than the effect of large ones. As an example, and

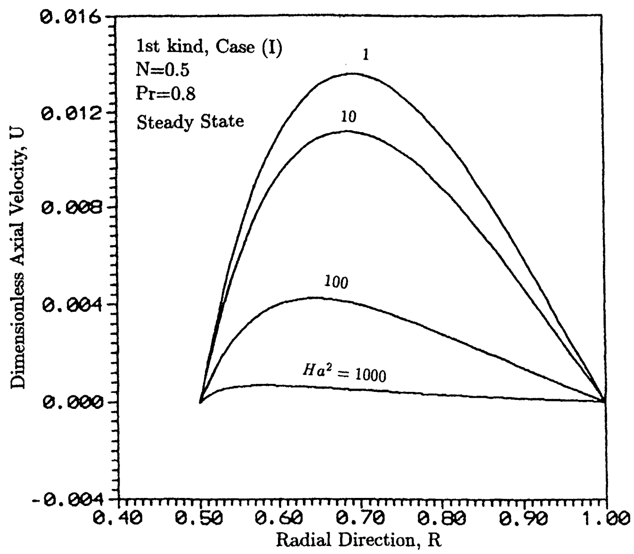

Fig. 4 Effect of *Ha* number on the velocity profile under steady conditions. First kind, Case (I).

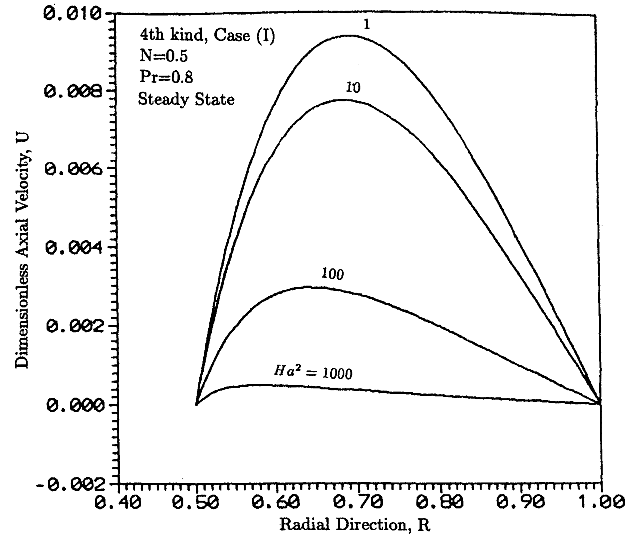

Fig. 5 Effect of *Ha* number on the velocity profile under steady conditions. Fourth kind, Case (I).

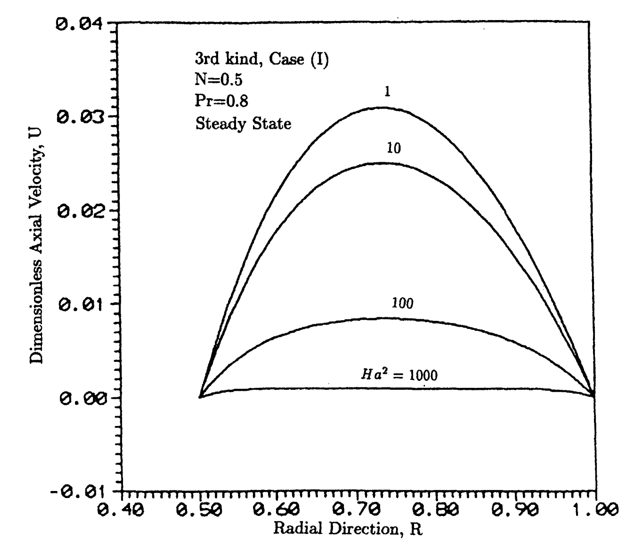

Fig. 6 Effect of *Ha* number on the velocity profile under steady conditions. Third kind, Case (I).

for the first kind of case (I), increasing *Ha* from 1 to 10 reduces the maximum velocity from 0.013 to 0.004. On the other hand, increasing *Ha* from 10 to $\sqrt{1000}$ reduces the maximum velocity from 0.004 to 0.001. The transient behavior of the volumetric flow rate is shown in Fig. 7 for cases (I) and (O) of the first kind fundamental solution. It is clear from this figure that case (O) has a higher volumetric flow rate than case (I). Because of the larger surface area of the outer wall, the heat transfer in case (O) is higher than that in case (I). Higher heat transfer in case (O) leads to a higher buoyancy driving force, and then yields a higher volumetric flow rate.

The variation of *Nu*, under steady conditions, with the annulus radius ratio is shown in Fig. 8 for cases (I) and (O) and for the first and fourth kinds of fundamental solutions. As shown from this figure, both kinds have the same *Nu* at steady-state conditions. This does not imply that these two kinds share the same velocity, volumetric flow rate, and temperature under steady conditions. Also, it is clear that case (I) has a higher *Nu* than case (O). Because of its larger surface area, the drifted flow from the boundary layer of the annulus outer wall is larger than that drifted from the boundary layer of the inner wall. As a result, the velocity profile will be shifted toward the inner wall if both walls have the same thermal boundary conditions. The higher velocity near the inner wall boundary layer enhances the heat transfer and yields higher *Nu*. However, and

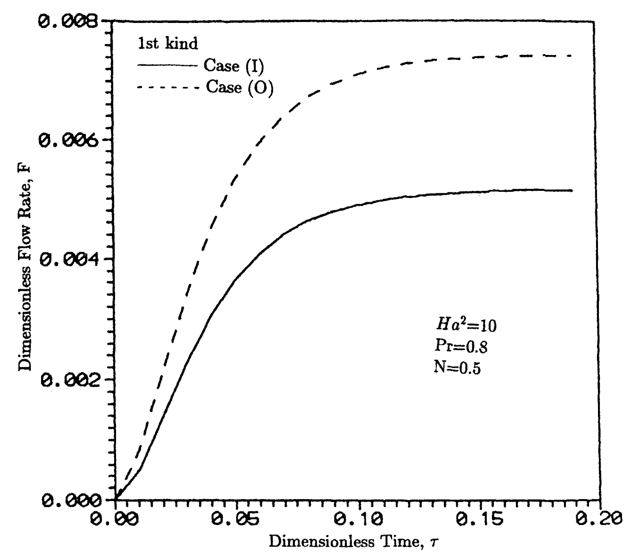

Fig. 7 Transient volumetric flow rate.

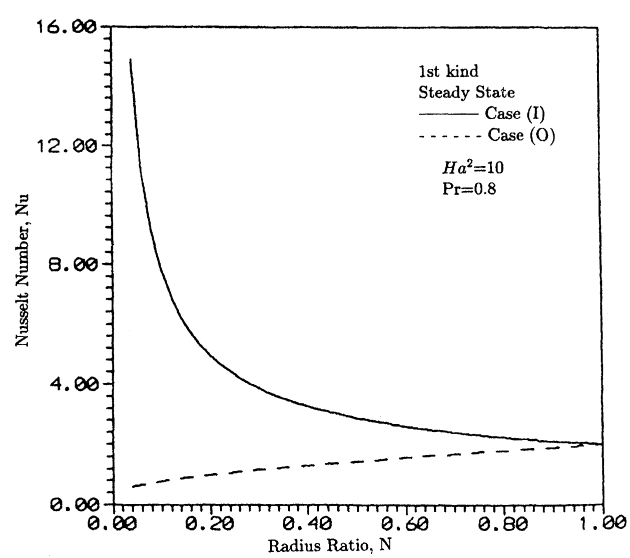

Fig. 8 Effect of the annulus radius ratio on *Nu*.

because of its larger surface area, the outer wall may have a higher total heat transfer and as a result, a higher volumetric flow rate. Also, and as N increases, both surfaces attain the same Nu because the configuration approaches that of the flow between two parallel plates and the gap width diminishes. The same figure shows that Ha has no effects on Nu for the first, third, and fourth kinds. This may be concluded directly from the governing Eqs. (1) and (2). For the mentioned three kinds of solutions, we have $\partial\theta/\partial Z = \partial P/\partial Z = 0$. As a result, θ is found from Eq. (2) without solving the momentum equation. This implies that Ha, which appears in Eq. (1), has no effect on Nu, which depends only on θ. The conclusion that Ha has no effect on Nu for the three kinds of solutions is valid under both transient and steady conditions. The effect of Ha on Nu is predicted to appear in the results of the second kind, which are not obtained in this study.

There is no need to plot the Nu behavior, under a steady condition, for the third kind of solution. For this the Nu is zero for cases (I) and (O). This behavior is predicted because for the third kind of solution one wall is insulated and the other is maintained at $\theta = 1$. As a result, and under steady conditions, the fluid temperature in the fully developed region is 1. This implies that there is no heat transfer between the heating wall and the fluid.

Fully Developed Natural Convection in Open-Ended Vertical Tube

Fundamental solutions for transient MHD fully developed natural convection flow in a vertical tube can be considered as a special case of that obtained for the annuli, but with $N = 0$. However, two fundamental cases are only possible in the case of tube. These are as follows.

Fundamental solution of the second kind, case (O):

$$\left.\frac{\partial\theta}{\partial R}\right|_{R=0} = 0, \qquad \left.\frac{\partial\theta}{\partial R}\right|_{R=1} = 1, \qquad \tau > 0 \tag{90}$$

Fundamental solution of the third kind, case (O):

$$\left.\frac{\partial\theta}{\partial R}\right|_{R=0} = 0, \qquad \theta(\tau, 1) = 1, \qquad \tau > 0 \tag{91}$$

The solution of the second case is obtained as follows. The eigenvalues λ_n are the roots of

$$J_0(\lambda_n) = 0 \tag{92}$$

Also, the temperature and velocity profiles are given as

$$\theta(\tau, R) = 1 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2\tau} J_0(\lambda_n R) \tag{93}$$

$$U(\tau, R) = \sum_{m=1}^{\infty} B_m\{1 - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\} J_0(\beta_m R) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\{e^{-\lambda_n^2\tau} - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\} J_0(\beta_m R) \tag{94}$$

where

$$A_n = -\frac{2}{\lambda_n J_1(\lambda_n)}, \qquad B_m = \frac{1}{8J_1(\beta_m)[\beta_m^3 + \beta_m Ha^2]} \tag{95}$$

$$B_{mn} = \begin{cases} 0 & n \neq m \\ -1/\{8J_1(\beta_m)[\beta_m^3 + \beta_m Ha^2 - (\beta_m^3/Pr)]\} & n = m \end{cases} \tag{96}$$

and the eigenvalues β_m are the roots of

$$J_0(\beta_m) = 0 \tag{97}$$

Also, the volumetric flow rate, mixing cup temperature, and local Nusselt number are given as

$$F(\tau) = \sum_{m=1}^{\infty} C_m\{1 - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\} - \sum_{m=1}^{\infty} C'_m\{e^{-\beta_m^2\tau} - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\} \tag{98}$$

$$\theta_m(\tau) = \frac{1}{F}\left\{\sum_{m=1}^{\infty} C_m\{1 - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\}(1 - e^{-\beta_m^2}) - \sum_{m=1}^{\infty} C'_m\{e^{-\beta_m^2\tau} - \exp[-Pr(\beta_m^2 + Ha^2)\tau]\}(1 - e^{-\beta_m^2})\right\} \tag{99}$$

The local Nusselt number for a UWT is given by

$$Nu = \pm 4 \sum_{n=1}^{\infty} e^{-\lambda_n^2\tau} \tag{100}$$

where

$$C_m = \frac{1}{4(\beta_m^4 + \beta_m^2 Ha^2)}, \qquad C'_m = \frac{1}{4(\beta_m^4 + \beta_m^2 Ha^2 - \beta_m^4)} \tag{101}$$

and the minus and plus signs apply, respectively, when there is heating, and vice versa when there is cooling.

Conclusions

Using Green's function method, analytical solutions for transient MHD fully developed upward (heating) or downward (cooling) natural convection velocity and temperature profiles in open-ended vertical concentric annuli and tube have been obtained. These solutions correspond to four fundamental boundary conditions obtained by combining each of the two conditions of having one boundary maintained at UHF or at UWT with each of the conditions that the opposite boundary is kept adiabatic or isothermal at the inlet fluid temperature. Expressions for transient fully developed volumetric flow rate, mixing cup temperature, and local Nusselt number are presented for each considered case. Such fully developed values are approached, in a given annulus, when the height to gap width ratio (l/b) is sufficiently large. These values represent the limiting conditions and provide analytical checks on numerical solutions for transient developing flows.

Once a developing natural convection flow reaches a state of full development, in a given annulus, the volumetric flow rate reaches its upper value; any further increase in the annulus height would not produce an increase in the volumetric flow rate. Moreover, for cases with an isothermal boundary, in a given annulus, the Nusselt number reaches its lower limiting value while the mixing cup temperature reaches its upper limiting value and all remain constant spacewise, but vary with time, irrespective of any further increase in the channel height. However, for cases with two UHF boundary conditions, in a given annulus, we are unable to get a closed-form solution.

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